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Tunneling in Magnetic Fields

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Tunneling effects in external magnetic fields are discussed in the model problem of a charged particle on a vertical rotating circle in a uniform gravitational field. The magnetic fields used are the ones for the monopole and a certain solution that arises in the context of the classical Weinberg–Salam model. Using instanton methods and the Gaussian approximation, various consequences regarding the tunnelling amplitudes and the level splittings in the presence of magnetic fields are obtained.

1. INTRODUCTION

In an earlier paper Kar (1992) discussed quantum tunnelling in the problem of a particle on a vertical rotating circle in a gravitational field, using instanton methods. There it was shown that for $\omega > \omega_0$ [ω is the angular frequency of rotation of the circle, $\omega_0 = (g/r)^{1/2}$, g is the acceleration due to gravity, and r is the radius of the circle] the apparent degeneracy in the minima of the potential is lifted due to tunnelling phenomena. Using the Gaussian approximation and the dilute gas sum, the splitting was evaluated. A correspondence between the particle on a rotating circle (PORC), the particle on a circle (POC), and the sine-Gordon (SG) and double-sine-Gordon (DSG) field theories was also discussed. Known solitary wave solutions of the $(1+1)$-dimensional DSG theory were used to write down the relevant instantons in PORC.

The present paper, as a sequel to Kar (1992), examines the effects of magnetic fields on tunnelling amplitudes and level splitting. Previous work related to tunneling in the presence of magnetic fields has been primarily on two-dimensional systems (Freed and Harvey, 1989; Jain and Kivelson,

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1988). Our problem is essentially one dimensional. We assume certain special magnetic fields in the presence of which the Lagrangian retains the same functional form. The net effect of the presence of such magnetic fields become encoded in the parameter a present in the potential term. However, our choice of these magnetic fields is not entirely arbitrary. Interestingly, they turn out to be the ones that arise for the magnetic monopole and in the context of certain classical solutions of the Weinberg–Salam model. Detailed discussion regarding them are discussed in the appendix. They are also obtained in Soni (1980) and Ryder (1985).

After a discussion of the classical mechanical problem in Section 2 we move on to the instanton solutions and Euclidean actions for the various cases. This comprises Section 3. In Section 4 we use the Gaussian approximation and the dilute gas sum to evaluate the tunneling amplitudes and level splitting in the presence of magnetic fields. Using a simple argument crucially dependent on the form invariance of the Lagrangian in the presence of magnetic fields, we show how the tunneling causes changes for the various cases. Finally in Section 5 we summarize our results.

2. THE CLASSICAL MECHANICAL PROBLEM IN EXTERNAL MAGNETIC FIELDS

In the absence of a magnetic field, the Lagrangian for a particle moving on a vertical circle rotating about the vertical axis in a gravitational field can be written as

$$L = \frac{1}{2} mr^2\dot{\vartheta}^2 - \frac{1}{2} m\omega^2 r^2 \left[\left(\cos\vartheta + \frac{g}{\omega^2 r}\right)\right]^2; \qquad \omega > \omega_0 \tag{1a}$$

or

$$L' = \frac{1}{2} mr^2\dot{\vartheta}^2 - \frac{1}{2} m\omega^2 r^2(\cos\vartheta + 1) \times \left(\cos\vartheta - 1 + 2\frac{g}{\omega^2 r}\right); \qquad \omega < \omega_0 \tag{1b}$$

Here ϑ is the only generalized coordinate, ω is the angular frequency of rotation of the circle, r is the radius of the circle, m is the mass of the particle, and g is the acceleration due to gravity. The reason for writing two different Lagrangians for the ranges $\omega > \omega_0$ and $\omega < \omega_0$ is discussed in Kar (1992). It is basically to allow us to take the $\omega \to 0$ limit in equation (1b).

We first deal with the case in which $\omega > \omega_{\text{crit}}$, where ω_{crit} is the value of ω beyond which the pair of degenerate minima appear in the effective potential function. These minima are separated by two kinds of barriers at $\vartheta = 0$ and $\vartheta = \pi$.

In the presence of a magnetic field the Lagrangian can be written using the standard form

$$L = \frac{1}{2} m|\mathbf{v}|^2 + q\mathbf{A} \cdot \mathbf{v} - V(\mathbf{x}) \tag{2}$$

The vector potentials for the two magnetic fields are given as follows ($\mathbf{A}_1$ denotes the potential for the field that arises in the context of classical solutions in the Weinberg–Salam model. $\mathbf{A}_2$ is the vector potential for the magnetic monopole):

$$\mathbf{A}_1 = \frac{\eta_1 \sin \vartheta}{r} \hat{e}_\varphi \tag{3a}$$

$$\begin{aligned} \mathbf{A}_2 &= \frac{\eta_2(1 - \cos \vartheta)}{r \sin \vartheta} \hat{e}_\varphi \qquad \text{(region excluding S pole)} \\ &= \frac{-\eta_2(1 + \cos \vartheta)}{r \sin \vartheta} \hat{e}_\varphi \qquad \text{(region excluding N pole)} \end{aligned} \tag{3b}$$

where η_1 and η_2 are two constants.

This construction for $\mathbf{A}_2$ is due to Wu and Yang (1975).

Using (2), we can write down the Lagrangians for the problem at hand. These are

$$L_1 = \frac{1}{2} mr^2\dot{\vartheta}^2 - \frac{1}{2} mr^2(\omega^2 + \beta_1 \omega)\left(\cos \vartheta + \frac{\omega_0^2}{\omega^2 + \beta_1 \omega}\right)^2 \tag{3}$$

$$L_2 = \frac{1}{2} mr^2\dot{\vartheta}^2 - \frac{1}{2} mr^2\omega^2\left(\cos \vartheta + \frac{\omega_0^2 + \beta_2 \omega}{\omega^2}\right)^2 \tag{4}$$

where

$$\beta_1 = \frac{2q\eta_1}{mr^2}, \qquad \beta_2 = \frac{q\eta_2}{mr^2}; \qquad q > 0 \tag{5}$$

Henceforth, we shall use $\beta_1 = \beta_2 = \beta$ and $\beta/\omega_0 = 1$. Also ω_0^2/ω^2 will be defined as a. The potentials in (1), (3), and (4) with the simplification turn out to be

$$V_{\text{eff0}}(\vartheta) = \frac{1}{2} mr^2\omega_0^2 \frac{1}{a}(\cos \vartheta + a)^2 \tag{6}$$

$$V_{\text{eff1}}(\vartheta) = \frac{1}{2} mr^2\omega_0^2 \frac{1}{k_1}(\cos \vartheta + k_1)^2 \tag{7}$$

$$V_{\text{eff2}}(\vartheta) = \frac{1}{2} mr^2\omega_0^2 \frac{1}{a}(\cos \vartheta + k_2)^2 \tag{8}$$

where

$$k_1 = \frac{a}{\sqrt{a} + 1}, \qquad k_2 = a + \sqrt{a} \tag{9}$$

At this point, it is important to mention an interesting fact that results when we have the monopole sitting at the center of the circle. Since $\omega_0 = (g/r)^{1/2}$, we can use $\omega_0 = 0$ as a method by which we can switch off the gravitational field. Without any magnetic field, this implies a simple cosine-squared potential and the barriers are at $\vartheta = \pi$, 2π (of equal height) with the minima at $\vartheta = \pi/2$, $\vartheta = 3\pi/2$. However, in the presence of a magnetic field, the potential takes the form

$$V_{\text{eff3}}(\vartheta) = \frac{1}{2} m r^2 \omega^2 (\cos\vartheta + \beta/\omega)^2 \tag{10}$$

For $\beta < \omega$ we have the same situation as we had when only a gravitational field was present.

Let us now analyze the maxima–minima structure of the effective potential in some detail. We have

$$\begin{aligned}
V_{\text{eff0}}(\vartheta):\quad & \vartheta = 0, \pi \text{ (maxima)} \\
& \vartheta = \cos^{-1}(-a), \qquad 2\pi - \cos^{-1}(-a) \text{ (minima)} \\
V_{\text{eff1}}(\vartheta):\quad & \vartheta = 0, \pi \text{ (maxima)} \\
& \vartheta = \cos^{-1}(-k_1), \qquad 2\pi - \cos^{-1}(-k_1) \text{ (minima)} \\
V_{\text{eff2}}(\vartheta):\quad & \vartheta = 0, \pi \text{ (maxima)} \\
& \vartheta = \cos^{-1}(-k_2), \qquad 2\pi - \cos^{-1}(-k_2) \text{ (minima)}
\end{aligned}$$

We now compare the effective potential in the presence of magnetic fields with the case where no magnetic field is present. The plots of $V_{\text{eff}}(\vartheta)$ versus ϑ for various cases are shown in Fig. 1.

For $V_{\text{eff1}}(\vartheta)$ the critical frequency ω_{01} beyond which the degenerate minima appear is given as

$$\omega_{01} = (\omega_0^2 + \beta_1^2/4)^{1/2} - \beta_1/2 \tag{11}$$

which is less than ω_0. In the case when $\beta_1 = \beta_2 = \omega_0$, $\omega_{01} = 0.618\omega_0$.

With $V_{\text{eff2}}(\vartheta)$, the critical frequency ω_{02} is given as

$$\omega_{02} = (\omega_0^2 + \beta_2^2/4)^{1/2} + \beta_2/2 \tag{12}$$

This is greater than ω_0 and is equal to $1.618\omega_0$ for $\beta_1 = \beta_2 = \omega_0$.

In the case of $V_{\text{eff1}}(\vartheta)$ the minima shift toward $\vartheta = 0$ (2π) as compared to the case without **B**. The opposite phenomena occur for $V_{\text{eff2}}(\vartheta)$. In this case both minima shift toward $\vartheta = \pi$. The barrier heights at $\vartheta = 0$ and $\vartheta = \pi$ change in the following way. For $V_{\text{eff1}}(\vartheta)$ the $\vartheta = 0$ barrier as well as the $\vartheta = \pi$ barrier can increase or decrease by the same amount as compared to the case with no magnetic field. If $\omega > \omega_0$, then there is an increase. If $\omega_{01} < \omega < \omega_0$, then there may be an increase or a decrease depending on

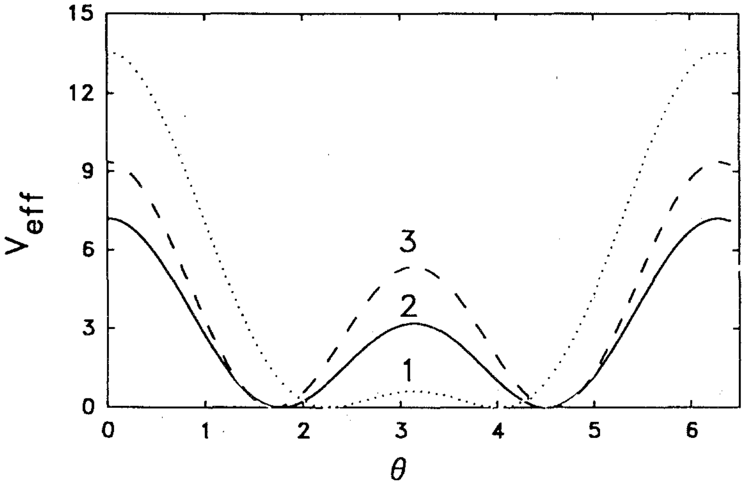

Fig. 1. Comparison of the potentials given in (1) equation (6); (2) equation (7); and (3) equation (8). The value of a chosen is 0.2.

whether $(\omega_0^2/\omega^2)\omega_0^2/(\omega_1^2+\beta\omega)$ or $a^2/(1+\sqrt{a})$ is less than or greater than 1. However for $0<a<1$, $a^2/(1+\sqrt{a})$ is always less than 1. Hence even for $\omega_{01}<\omega<\omega_0$ the barrier heights increase. For $V_{\text{eff2}}(\vartheta)$ the $\vartheta=0$ barrier decreases and the $\vartheta=\pi$ barrier increases. However, the amount of increase is not equal to the amount of decrease—in fact, the former is more than the latter for all values of $0<a<0.382$ $(0<k_1<1)$.

In the case when the effective potential does not exhibit a pair of degenerate vacua, the Lagrangian in the presence of external magnetic fields can be written as follows [these are constructed in such a way that they look similar in form to equation (1b); the subscripts 1 and 2 refer to the magnetic fields mentioned before]:

$$L_1'=\frac{1}{2}mr^2\dot{\vartheta}^2-\frac{1}{2}mr^2\frac{\omega_0^2}{k_1}[(\cos\vartheta+1)(\cos\vartheta-1+2k_1)] \tag{13}$$

$$L_2'=\frac{1}{2}mr^2\dot{\vartheta}^2-\frac{1}{2}mr^2\frac{\omega_0^2}{a}[(\cos\vartheta+1)(\cos\vartheta-1+2k_2)] \tag{14}$$

Therefore in order to compare the tunneling amplitudes and the level splittings in these cases with those for the case of no magnetic field we have to take values of $a>1.618$ for L', L_1' and $a>1$ for L', L_2'.

Also the only barrier at $\vartheta=0$ is affected in the following way in the presence of magnetic fields. For the potential in L_1' it remains unaltered, whereas for the potential in L_2' it decreases.

Qualitatively, therefore, we expect a decrease in the tunneling amplitude as well as the level splitting for $V_{\text{eff1}}(\vartheta)$. For $V_{\text{eff2}}(\vartheta)$ they should also decrease. Similar results can be derived for the case when the potential does not exhibit a pair of degenerate minima. In the next two sections, we use semiclassical arguments to arrive at these conclusions.

As a passing remark, we mention that the $(0+1)$-dimensional Lagrangians mentioned here are essentially one-dimensional analogs to the DSG theory in $1+1$ dimensions. The Lagrangian density for the DSG theory is given as

$$\mathscr{L} = \frac{1}{2}(a_\mu \phi)(a^\mu \phi) - \frac{1}{2\rho_1}(\cos\phi + \rho_2) \tag{15}$$

Therefore in constructing the instantons one can either use the solitons of (7) or directly integrate the equation of motion in Euclidean time.

3. THE INSTANTONS

The instanton solutions and the Euclidean actions for the Lagrangians given in (1), (3), and (4) are listed below. The Euclidean actions are denoted by S_{E_i}, where i is a subscript corresponding to the relevant $\vartheta_i(\tau)$.

(a) *Lagrangian L*

(i) Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_1(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tanh\frac{(1-a^2)^{1/2}}{2}\omega\tau\right] \tag{16}$$

$$S_{E_1} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-a^2}{a}\right)^{1/2} - 2\sqrt{a}\tan^{-1}\left(\frac{1+a}{1-a}\right)^{1/2}\right] \tag{17}$$

(ii) Instanton/anti-instanton across the $\vartheta = \pi$ barrier:

$$\vartheta_2(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \coth\frac{(1-a^2)^{1/2}}{2}\omega\tau\right] \tag{18}$$

$$S_{E_2} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-a^2}{a}\right)^{1/2} - 2\sqrt{a}\tan^{-1}\left(\frac{1-a}{1+a}\right)^{1/2}\right] \tag{19}$$

(b) *Lagrangian L_1*

(i) Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_3(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+k_1}{1-k_1}\right)^{1/2} \tanh\frac{(1-a^2)^{1/2}}{2}(1+\sqrt{a})^{1/2}\omega\tau\right] \tag{20}$$

$$S_{E_3} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-k_1^2}{k_1}\right)^{1/2} + 2k_1^{1/2}\tan^{-1}\left(\frac{1+k_1}{1-k_1}\right)^{1/2}\right] \tag{21}$$

(ii) Instanton/anti-instanton across the $\vartheta = \pi$ barrier:

$$\vartheta_4(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+k_1}{1-k_1}\right)^{1/2} \coth \frac{(1-a^2)^{1/2}}{2}(1+\sqrt{a})^{1/2}\omega\tau\right] \quad (22)$$

$$S_{E_4} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-k_1^2}{k_1}\right)^{1/2} - 2k_1\tan^{-1}\left(\frac{1-k_1}{1+k_1}\right)^{1/2}\right] \quad (23)$$

(c) *Lagrangian* L_2

(i) Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_5(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+k_2}{1-k_2}\right)^{1/2} \tanh \frac{(1-k_2^2)^{1/2}}{2}\omega\tau\right] \quad (24)$$

$$S_{E_5} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-k_2^2}{a}\right)^{1/2} + \frac{2k_2}{\sqrt{a}}\tan^{-1}\left(\frac{1+k_2}{1-k_2}\right)^{1/2}\right] \quad (25)$$

(ii) Instanton/anti-instanton across the $\vartheta = \pi$ barrier:

$$\vartheta_6(\tau) = \pm 2\tan^{-1}\left[\left(\frac{1+k_2}{1-k_2}\right)^{1/2} \coth \frac{(1-k_2^2)^{1/2}}{2}\omega\tau\right] \quad (26)$$

$$S_{E_6} = 2mg^{1/2}r^{3/2}\left[\left(\frac{1-k_2^2}{a}\right)^{1/2} - \frac{2k_2}{\sqrt{a}}\tan^{-1}\left(\frac{1-k_2}{1+k_2}\right)^{1/2}\right] \quad (27)$$

(d) *Lagrangian* L'

Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_7(\tau) = \pm 2\tan^{-1}\left[\left(\frac{a}{a-1}\right)^{1/2} \sinh \frac{(a-1)^{1/2}}{a}\omega_0\tau\right] \quad (28)$$

$$S_{E_7} = 4mg^{1/2}r^{3/2}\left[\sqrt{a}\tan^{-1}\frac{1}{(a-1)^{1/2}} + \left(\frac{a-1}{a}\right)^{1/2}\right] \quad (29)$$

(e) *Lagrangian* L_1'

Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_8(\tau) = \pm 2\tan^{-1}\left[\left(\frac{k_1}{k_1-1}\right)^{1/2} \sinh \frac{(k_1-1)^{1/2}}{k_1}\omega_0\tau\right] \quad (30)$$

$$S_{E_8} = 4mg^{1/2}r^{3/2}\left[k_1^{1/2}\tan^{-1}\frac{1}{(k_1-1)^{1/2}} + \left(\frac{k_1-1}{k_1}\right)^{1/2}\right] \quad (31)$$

(f) *Lagrangian* L_2'

Instanton/anti-instanton across the $\vartheta = 0$ barrier:

$$\vartheta_9(\tau) = \pm 2 \tan^{-1}\left[\left(\frac{k_2}{k_2 - 1}\right)^{1/2} \sinh \frac{(k_2 - 1)^{1/2}}{a} \omega_0 \tau\right] \tag{32}$$

$$S_{E_9} = 4mg^{1/2} r^{3/2}\left[\frac{k_2}{\sqrt{a}} \tan^{-1} \frac{1}{(k_2 - 1)^{1/2}} + \left(\frac{k_2 - 1}{a}\right)^{1/2}\right] \tag{33}$$

4. TUNNELING AMPLITUDES, LEVEL SPLITTING

Our analysis of the tunneling amplitude and the level splitting is based on the Gaussian approximation to the path-integral representation of the amplitude and the dilute gas sum. Using these, we can write down the relevant expressions for the amplitudes and the level splitting formulas. A word here about the notation. T_i, Δ_i and T_i', Δ_i' $(i = 0, 1, 2)$ denote the total tunneling amplitude and the magnitude of the splitting, respectively. The prime refers to the case when the effective potential has a single barrier at $\vartheta = \pi$. The subscript i represents the three cases of no magnetic field and fields generated by $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. These T_i, Δ_i are given as

$$T_0 = \left(\frac{\alpha_0}{\pi\hbar}\right)^{1/2} (\omega_0 \tau / \sqrt{a}) \exp(-\alpha_0 \omega_0 \tau / 2\sqrt{a})$$
$$\times [K_1(a)(S_{E_1})^{1/2} \exp(-S_{E_1}/\hbar) + K_2(a)(S_{E_2})^{1/2} \exp(-S_{E_2}/\hbar)] \tag{34}$$

$$T_1 = (\alpha_1/\pi\hbar)^{1/2} (\omega_0 \tau / \sqrt{k_1}) \exp(-\alpha_0 \omega_0 \tau / 2\sqrt{k_1})$$
$$\times [K_1(k_1)(S_{E_3})^{1/2} \exp(-S_{E_3}/\hbar) + K_2(k_1)(S_{E_4})^{1/2} \exp(-S_{E_4}/\hbar)] \tag{35}$$

$$T_2 = (\alpha_2/\pi\hbar)^{1/2} (\omega_0 \tau / \sqrt{a}) \exp(-\alpha_0 \omega_0 \tau / 2\sqrt{a})$$
$$\times [K_1(k_2)(S_{E_5})^{1/2} \exp(-S_{E_5}/\hbar) + K_2(k_2)(S_{E_6})^{1/2} \exp(-S_{E_6}/\hbar)] \tag{36}$$

$$\Delta_0 = (2\hbar\omega_0/\sqrt{a})[K_1(a)(S_{E_1})^{1/2} \exp(-S_{E_1}/\hbar)$$
$$+ K_2(a)(S_{E_2})^{1/2} \exp(-S_{E_2}/\hbar)] \tag{37}$$

$$\Delta_1 = (2\hbar\omega_0/\sqrt{k_1})[K_1(k_1)(S_{E_3})^{1/2} \exp(-S_{E_3}/\hbar)$$
$$+ K_2(k_1)(S_{E_4})^{1/2} \exp(-S_{E_4}/\hbar)] \tag{38}$$

$$\Delta_2 = (2\hbar\omega_0/\sqrt{a})[K_1(k_2)(S_{E_5})^{1/2} \exp(-S_{E_5}/\hbar)$$
$$+ K_2(k_2)(S_{E_6})^{1/2} \exp(-S_{E_6}/\hbar)] \tag{39}$$

where α_0, α_1, and α_2 are given as

$$\alpha_0^2 = mr^2(1 - a^2), \qquad \alpha_1^2 = mr^2(1 - k_1^2), \qquad \alpha_2^2 = mr^2(1 - k_2^2) \tag{40}$$

T'_i and Δ'_i are given by

$$T'_0 = \left(\frac{\alpha'_0}{\pi\hbar}\right)^{1/2} \frac{\omega_0 \tau}{\sqrt{a}} \exp\left(-\frac{\alpha_0 \omega_0 \tau}{2\sqrt{a}}\right)\left[K_3(a)(S_{E_7})^{1/2} \exp\frac{-S_{E_7}}{\hbar}\right] \quad (41)$$

$$T'_1 = \left(\frac{\alpha'_1}{\pi\hbar}\right)^{1/2} \frac{\omega_0 \tau}{\sqrt{k_1}} \exp\left(-\frac{\alpha_0 \omega_0 \tau}{2\sqrt{k_1}}\right)\left[K_3(k_1)(S_{E_8})^{1/2} \exp\frac{-S_{E_8}}{\hbar}\right] \quad (42)$$

$$T'_2 = \left(\frac{\alpha'_2}{\pi\hbar}\right)^{1/2} \frac{\omega_0 \tau}{\sqrt{a}} \exp\left(-\frac{\alpha_0 \omega_0 \tau}{2\sqrt{a}}\right)\left[K_3(k_2)(S_{E_9})^{1/2} \exp\frac{-S_{E_9}}{\hbar}\right] \quad (43)$$

$$\Delta'_0 = \frac{2\hbar\omega_0}{\sqrt{a}}\left[K_3(a)(S_{E_7})^{1/2} \exp\frac{-S_{E_7}}{\hbar}\right] \quad (44)$$

$$\Delta'_1 = \frac{2\hbar\omega_0}{\sqrt{k_1}}\left[K_3(k_1)(S_{E_8})^{1/2} \exp\frac{-S_{E_8}}{\hbar}\right] \quad (45)$$

$$\Delta'_2 = \frac{2\hbar\omega_0}{\sqrt{a}}\left[K_3(k_2)(S_{E_9})^{1/2} \exp\frac{-S_{E_9}}{\hbar}\right] \quad (46)$$

where α'_0, α'_1, α'_2 are given as

$$\alpha'^2_0 = mr^2(a-1), \qquad \alpha'^2_1 = mr^2(k_1 - 1), \qquad \alpha'^2_2 = mr^2(k_2 - 1) \quad (47)$$

Notice that we used the same functional forms K_1, K_2, K_3 for the ratio of determinants in the expressions above. This arises out of the form invariance of the Lagrangian in the presence of magnetic fields relative to the case without them. We now deal with the four cases separately:

(a) In the presence of the magnetic field generated by the vector potential $\mathbf{A}_1$ we find that the tunneling amplitude and the level splitting both decrease. This happens as follows. Assume a value $k_1 = \beta$ and substitute this in the expressions for T_1 and Δ_1. Then, using the same β for a in the expressions for T_0 and Δ_0, we find that

$$T_0(a=\beta) = T_1(k_1=\beta) \qquad \Delta_0(a=\beta) = \Delta_1(k_1=\beta) \quad (48)$$

However, $k_1 = \beta$ implies that the actual value of a involved in (9) is larger than k_1. This means that there is a one-to-one correspondence between T_0, Δ_0 and T_1, Δ_1 provided the value of a used to evaluate the former is the same as the value of k_1 used to evaluate the latter. Hence, in the presence of the magnetic field the splitting of the tunneling amplitude remains the same if the angular frequency of rotation is reduced in a certain way. Now if we keep ω or a fixed, say, at a value $a = \gamma$, we will have

$$T_0(a=\gamma) > T_1(a=\gamma), \qquad \Delta_0(a=\gamma) > \Delta_1(a=\gamma) \quad (49)$$

This follows from the fact that as a decreases, T_i and Δ_i decrease and they ultimately go to zero as $a \to 0$ ($\omega \to \infty$). Our analysis therefore proves our assertion that the tunneling amplitudes and splittings would decrease in the presence of the field due to $\mathbf{A}_1$. This is in conformity with the fact shown in Section 2 that the barrier heights at $\vartheta = 0$ and $\vartheta = \pi$ both increase relative to the case of no magnetic field.

(b) In the presence of a magnetic monopole we can see very clearly that the trick used in the previous case does not work. This happens because we cannot write $T_0(a = \beta) = T_2(k_2 = \beta)$. The expression for T_2 includes k_2 as well as a. Thus, if we use $k_2 = \beta$, we need to use an a which comes from the solution of the equation $a + \sqrt{a} = \beta$. On the other hand, if we use $a = \beta$ in T_0, we find no such situation. Therefore, we shall only give a partial answer to the question: what happens to the tunneling amplitude when a monopole is sitting at the center?

Assume a situation in which we can tune the frequency of rotation of the circle. Initially there is no monopole and we have $a = \beta$, say. Then we place the monopole and tune the frequency in such a way that now $k_2 = \beta$. The position of the minima therefore remain unaltered. But the barrier heights are altered. We shall see that the tunneling amplitude across the $\vartheta = 0$ barrier (T_{20}) decreases whereas the one across the $\vartheta = \pi$ barrier ($T_{2\pi}$) increases. Let us forget about the τ-dependent factors. These when analytically continued to real time give factors of the form ($e^{it}it$), which is a phase factor multiplied by t.

Define $\tilde{S}_{E_i} = \sqrt{a}\, S_{E_i}$ ($i = 1, 2, 5, 6$). Then

$$\frac{|T_{20}|}{|T_{00}|} = \left(\frac{a + \sqrt{a}}{a}\right)^{3/4} \times \left[\exp\left(\frac{\tilde{S}_{E_5}}{\hbar}\right)\right]^{1/(a+\sqrt{a})^{1/2} - 1/\sqrt{a}} \tag{50}$$

For $|T_{20}|/|T_{00}| < 1$ we require

$$\frac{\tilde{S}_{E_5}}{\hbar} > \frac{3/4 \ln[(a + \sqrt{a})/a]}{1/\sqrt{a} - 1/(a + \sqrt{a})^{1/2}} \tag{51}$$

Similarly

$$\frac{|T_{2\pi}|}{|T_{0\pi}|} = \left(\frac{a + \sqrt{a}}{a}\right)^{3/4} \times \left[\exp\left(\frac{\tilde{S}_{E_6}}{\hbar}\right)\right]^{1/(a+\sqrt{a})^{1/2} - 1/\sqrt{a}} \tag{52}$$

For $|T_{2\pi}|/|T_{0\pi}| > 1$ we require

$$\frac{\tilde{S}_{E_6}}{\hbar} < \frac{3/4 \ln[(a+\sqrt{a})/a]}{1/\sqrt{a} - 1/(a+\sqrt{a})^{1/2}} \tag{53}$$

Taking $2mg^{1/2}r^{3/2} \approx h$, we can check that both equations (51) and (53) are satisfied for $0.064 < a < 0.038$. For $0 < a < 0.064$ the sign of the inequality (53) is reversed. Hence one can say that in the presence of a monopole at the center the tunneling amplitude across the $\vartheta = 0$ barrier decreases, whereas it increases across the $\vartheta = \pi$ barrier. Of course we have tuned the frequency to a lower value in order to keep the positions of the minima fixed.

One cannot comment on the total tunneling amplitude without evaluating the preexponential factors. Nor can one conclude whether the splitting increases or decreases. A fuller understanding of the solution would have to await the solution of the relevant Schrödinger equation whose eigenvalues would determine the corresponding fluctuation determinant.

(c) In this case, where the potential exhibits a single maximum at $\vartheta = \pi$ (and minimum at $\vartheta = 0, 2\pi$) and the external magnetic field is generated by $\mathbf{A}_1$, one can proceed in the same way as in (a) and show that the tunneling amplitude as well as the level splitting decrease in the presence of the magnetic field.

(d) Finally, if the potential exhibits a single maximum and we have a monopole at the center, we can analyze the situation along the same lines as in (b). With $k_2 = \xi$ and $a = \xi$ in the expressions for T_2 and T_0', respectively, we get (with $\tilde{S}_{E_9} = \sqrt{a}\, \tilde{S}_{E_9}$)

$$\frac{|T_2'|}{|T_0'|} = \left(\frac{a+\sqrt{a}}{a}\right)^{3/4} \times \left[\exp\left(\frac{-\tilde{S}_{E_9}}{\hbar}\right)\right]^{-1/(a+\sqrt{a})^{1/2} + 1/\sqrt{a}} \tag{54}$$

Thus $T_2' < T_0'$ if

$$\frac{\tilde{S}_{E_9}}{\hbar} > \frac{3/4 \ln[(a+\sqrt{a})/a]}{1/\sqrt{a} - 1/(a+\sqrt{a})^{1/2}} \tag{55}$$

One can check that for all $a > 1$, equation (55) is satisfied. Thus we can say that

$$\Delta_0'(a = \xi) > \Delta_2'(k_2 = \xi) \tag{56}$$

Now $k_2 = \xi$ implies a similar value of a. Therefore, if we increase the frequency of rotation in such a way that the fluctuation determinants cancel out in the ratio, we can say that the ratios of the splittings and the tunneling amplitudes vary according to (54) and (55).

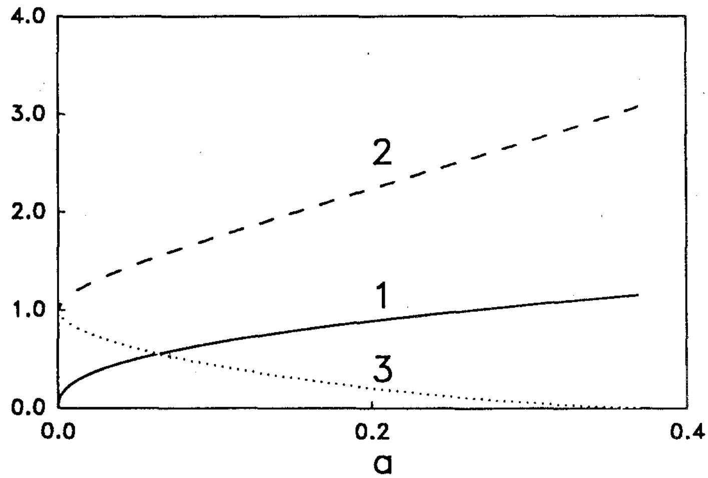

Fig. 2. The validity of equations (51) and (53). (1) The function containing the logarithm, (2) the function $\tilde{S}_{E_5}$, and (3) the function $\tilde{S}_{E_6}$.

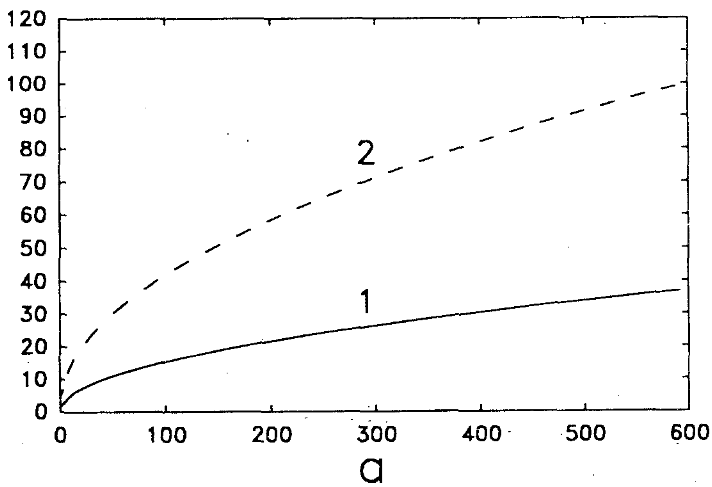

Fig. 3. The validity of equation (55). (1) The function containing the logarithm, (2) the function $\tilde{S}_{E_9}$.

Figures 2 and 3 show the validity of equations (51), (53), and (55).

5. SUMMARY AND CONCLUSION

The idea behind this paper was to explore the consequences of external magnetic fields on tunneling amplitudes and level splitting in the model problem of a charged particle on a vertical rotating circle in a uniform gravitational field. We have considered two magnetic fields, generated by the vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$. For the first one the analysis was simple. The crucial point was the fact that as $a \to k_1$ the Lagrangian in the absence of the magnetic field went over to that in the presence of it. At the classical level, the minima shift toward $\vartheta = 0\ (2\pi)$ and the barrier heights increase for the case in which $0 < k_1 < 1$. If $k_1 > 1$, then the only maximum is at $\vartheta = 0$ and the barrier height does not change. Quantum mechanical tunneling takes place, leading to a splitting in the degenerate minima for $0 < k_1 < 1$. The tunneling amplitude and level splitting both decreases relative to the case with no magnetic fields. For $k_1 > 1$, where tunneling is associated with the winding of the particle around the circle, there is a similar "splitting." In the same way as for $0 < k_1 < 1$, the tunneling amplitude and splitting are lower than in the case without magnetic field.

In the second case, where the magnetic field is due to a magnetic monopole sitting at the center, we have attempted to give a partial answer to the question regarding tunneling amplitudes and level splitting. Keeping the positions of the minima fixed by tuning the frequency, we have been able to show that the presence of the monopole is manifested in the decrease of the tunneling amplitude across the $\vartheta = 0$ barrier and an increase across the $\vartheta = \pi$ barrier for $0.064 < a < 0.38$. If $a < 0.064$ both the amplitudes decrease. For the case when $k_2 > 1$ a similar analysis shows a decrease in the tunneling amplitude and level splitting. Although we have derived certain consequences, the full understanding of the tunneling phenomena in this model problem requires an evaluation of the fluctuation or instanton determinant. This would give us explicit forms for K_1, K_2, and K_3 and we could then compute exactly the relevant amplitudes and splitting. Throughout our analysis we have tried to derive results which depend on the forms of K_1, K_2, and K_3. Whereas a knowledge of this is not necessary for the magnetic field generated by $\mathbf{A}_1$, it is absolutely necessary for the problem with the monopole. Attempts toward obtaining the eigenvalues of the relevant Schrödinger equation (necessary for evaluating the instanton determinant) are in progress and the results will be communicated in future publications.

APPENDIX

Let us discuss the magnetic fields we have used in some detail.

The first one, i.e., $\mathbf{A}_1$ arises in the context of classical solutions of the Weinberg–Salam model. The Lagrangian for this model can be written as

$$\mathscr{L} = \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - (D_\mu \phi)^\dagger (D^\mu \phi) - \lambda(|\phi|^2 - F^2) \right] \quad \text{(A1)}$$

We will look for a finite-energy configuration (FEC) of this model. This FEC will result in the magnetic field that we are considering.

We begin with an ansatz for the $SU(2)$ Higgs field in the following asymptotic form:

$$\phi = F\begin{bmatrix} \cos\vartheta \\ \sin\vartheta\, e^{i\psi} \end{bmatrix} \tag{A2}$$

where F is the vacuum expectation value of ϕ, determined from the minimization of the potential.

Now, to look for a FEC we must satisfy $D_i\phi|_{r\to\infty} = 0$ asymptotically. Notice that $\phi = \text{const}$, $\partial_i\phi|_{r\to\infty} \sim 1/r$, and therefore $A_\mu|_{r\to\infty} \sim 1/r$. Now, this condition is adequate to obtain the asymptotic gauge fields. The leading behavior of the field equations is

$$\begin{aligned} \partial_j F^a_{ij} + g\varepsilon^{abc} A^b_j F^c_{ij} &= -i\frac{g}{2}(\phi\dagger\tau^a D_i\phi - \text{c.c.}) \\ \partial_j F^0_{ij} &= -i\frac{g}{2}(\phi\dagger D_i\phi - \text{c.c.}) \end{aligned} \tag{A3}$$

It is clear that, asymptotically, the left-hand side goes nominally as $1/r^3$ and the right-hand side with single derivative as $1/r$. Therefore, to leading order, we can drop the LHS and impose $D_i\phi|_{r\to\infty} = 0$. Using the elegant prescription of Nambu (1977), we can solve for the asymptotic gauge fields (Soni, 1980)

$$\begin{aligned} gA^a_i &= 2(1-\eta)\frac{e_\phi}{r}\sin\vartheta\,(e_r\cos\vartheta + e_\vartheta\sin\vartheta)^a \\ &\quad - 2\frac{e_\phi}{r}\cos\vartheta\,(e_r\sin\vartheta - e_\vartheta\cos\vartheta)^a - 2\frac{e_\vartheta}{r}(e_\phi)^a \\ g'A^0_i &= 2\eta\frac{e_\phi}{r}\sin\vartheta \end{aligned} \tag{A4}$$

The electromagnetic tensor can then be extracted as

$$f^{\text{em}}_{\mu\nu} = -[g'F^a_{\mu\nu}[\phi\dagger\tau^a\phi/|\phi|^2] - gF^0_{\mu\nu}](g^2 + g'^2)^{1/2} \tag{A5}$$

Since

$$gF^a_{\mu\nu} = -\eta f_{\mu\nu}(\phi\dagger\tau^a\phi)/|\phi|^2 \tag{A6}$$

where

$$f_{\mu\nu} = -2i[\partial_\mu\phi\dagger\partial_\nu\phi - \partial_\nu\phi\dagger\partial_\mu\phi]/|\phi|^2 \tag{A7}$$

and

$$g'F^0_{\mu\nu} = \eta f_{\mu\nu} \tag{A8}$$

we have

$$f_{\mu\nu}^{\text{em}} = \eta[(g^2 + g'^2)^{1/2}/gg']f_{\mu\nu} = [\eta/e]f_{\mu\nu} \tag{A9}$$

Since $f_{\mu\nu}$ is Abelian, it is linear in the electromagnetic potential and we have

$$A_i^{\text{em}} = (2\eta/e)(e_\phi)_i(\sin\vartheta)/r = \eta_1[(\sin\vartheta)/r](e_\phi) \tag{A10}$$

Let us now look at the case for a magnetic monopole.

Consider a magnetic monopole of strength g at the origin. The magnetic field is given by

$$\bar{B} = (g/r^3)\bar{r} = -g\nabla(1/r) \tag{A11}$$

So,

$$\bar{\nabla}\cdot\bar{B} = 4\pi gr\,\delta^3(r) \tag{A12}$$

The total flux through a sphere surrounding the origin is

$$\phi = 4\pi r^2 B = 4\pi g \tag{A13}$$

The wave function for a particle of charge e in the field of this monopole is

$$\psi = |\psi|\exp\left[\frac{i}{h}(\bar{p}\cdot\bar{r} - Et)\right] \tag{A14}$$

In the presence of an electromagnetic field, $\bar{p}\to\bar{p}-(e\bar{A}/c)$, so

$$\psi \to \psi\exp\left(-\frac{ie}{\hbar c}\bar{A}\cdot\bar{r}\right) \tag{A15}$$

or the phase α changes by

$$\alpha \to \alpha - \frac{e}{\hbar c}\bar{A}\cdot\bar{r} \tag{A16}$$

Consider a closed path at fixed r, ϑ with ϕ ranging from 0 to 2π. The total change in phase is

$$\begin{aligned}\Delta\alpha &= \frac{e}{\hbar c}\int \bar{A}\cdot d\bar{l} \\ &= \frac{e}{\hbar c}\int(\text{curl }\bar{A})\cdot d\bar{S} \\ &= \frac{e}{\hbar c}\int \bar{B}\cdot d\bar{S} \\ &= \frac{e}{\hbar c}\times(\text{flux through a cap}) \\ &= \frac{e}{\hbar c}\phi(r,\vartheta)\end{aligned} \tag{A17}$$

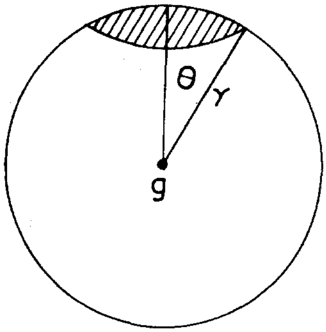

Fig. 4. A magnetic monopole sitting at the center of a sphere.

$\phi(r, \vartheta)$ is the flux through the cap defined by a particular r and ϑ. As ϑ is varied, the flux through the cap varies as shown in Fig. 4. As $\vartheta \to 0$, the loop shrinks to a point and the flux passing through the cap approaches zero: $\phi(r, \vartheta) = 0$.

As the loop is lowered over the sphere, the cap encloses more and more flux until at $\vartheta \to \pi$ we should have, from (A13),

$$\phi(r, \pi) = 4\pi g \tag{A18}$$

However, as $\vartheta \to \pi$, the loop has again shrunk to a point, so the requirement that $\phi(r, \pi)$ is finite entails, from (A17), that A is singular at $\vartheta = \pi$. This argument holds for all spheres of all possible radii, so it follows that A is singular along *the entire negative z axis*. This is known as the Dirac string.

Now it is clear from (A12) that if $\bar{B} = \text{curl } \bar{A}$ and A is regular, then $\text{div } \bar{B} = 0$, and no magnetic charge may exist. From the argument above, A is constructed by considering the pole as the endpoint of a string of magnetic dipoles whose other end is at infinity. This gives

$$A_r = A_\vartheta = 0, \qquad A_\phi = \frac{g}{r}\frac{(1 - \cos\vartheta)}{\sin\vartheta} \tag{A19}$$

A is clearly singular along $r = -z$. If, on the other hand the Dirac string had been chosen to be along $r = z$, we would have had

$$A_r = A_\vartheta = 0, \qquad A_\phi = \frac{-g}{r}\frac{1 - \cos\vartheta}{\sin\vartheta} \tag{A20}$$

So, with $g = \eta_2$ we can write

$$\begin{aligned} A &= \frac{\eta_2(1 - \cos\vartheta)}{r \sin\vartheta}\hat{e}_\varphi \qquad \text{(region excluding S pole)} \\ &= \frac{-\eta_2(1 + \cos\vartheta)}{r \sin\vartheta}\hat{e}_\varphi \qquad \text{(region excluding N pole)} \end{aligned} \tag{A21}$$

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